2020

MATHEMATICS — HONOURS

Paper : CC-6

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

1. Choose the correct alternative and justify your answer (1 mark for correct answer and 1 mark for justification). (1+1)×10=20

(a) The ring of matrices $S=\left\{\begin{pmatrix} a & b \\ 2b & a \end{pmatrix} : a, b \in \mathbb{R}\right\}$ contains

(i) no divisors of zero with unity (ii) divisors of zero with unity

(iii) no divisors of zero without unity (iv) divisors of zero without unity.

(b) Which of the following ring is not an integral domain?

(i) $(\mathbb{Z},+,\cdot)$ (ii) $(\mathbb{Z}_6,+,\cdot)$

(iii) $(\mathbb{Z}_5,+,\cdot)$ (iv) $(\mathbb{Z}_{13},+,\cdot)$.

(c) Let R be a ring where $R=(\mathbb{Z}_8,+,\cdot)$. Then char (R) is

(i) 0 (ii) 2

(iii) 8 (iv) 4.

(d) Let $S=\{a+b\omega : a, b \in \mathbb{R}\}$, where ω is an imaginary cube root of unity. Then

(i) S is a subring but not a subfield of $\mathbb{C}$ (ii) S is a subfield of $\mathbb{C}$

(iii) S is not a subring of $\mathbb{C}$ (iv) None of these.

(e) Let R be a ring with unity element e and let $\phi : R \to R'$ be a ring homomorphism. If $\phi(e) = 0$, where 0 is the zero element of the ring $R'(\neq R)$. Then ker ϕ is

(i) R (ii) R'

(iii) an empty set (iv) None of these.

(f) Which one of the following ideals is not a prime ideal?

(i) $2\mathbb{Z}$ in the ring $\mathbb{Z}$ (ii) $4\mathbb{Z}$ in the ring $2\mathbb{Z}$

(iii) The ideal $\{0\}$ in $\mathbb{Z}$ (iv) The ideal $\{0\}$ in $\mathbb{R}$.

(g) The dimension of the vector space $\mathbb{C}$ over $\mathbb{R}$ is

(i) infinite (ii) 1

(iii) 2 (iv) none of these.

(h) Let V be a vector space of all 3×3 real matrices over the field $\mathbb{R}$. Then the dimension of the subspace W consisting of all symmetric matrices is

(i) 4 (ii) 5

(iii) 6 (iv) 3.

(i) Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be a Linear mapping. Then nullity of T+ rank of T is

(i) 1 (ii) 2

(iii) 3 (iv) 4.

(j) If λ is an eigenvalue of a real skew symmetric matrix, then $\left|\frac{1-\lambda}{1+\lambda}\right|$ is

(i) 4 (ii) 1

(iii) 3 (iv) 2.

Unit - I

2. Answer ***any five*** questions : 5×5=25

(a) (i) Show that in a commutative ring R with unity, if M is a maximal ideal of R, then the quotient ring R/M is a field.

(ii) Correct or justify : A maximal ideal in a commutative ring may not be a prime ideal. 3+2

(b) State fundamental theorem of ring homomorphism. Let $R = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in \mathbb{Z} \right\}$ and $\varphi : R \to \mathbb{Z}$ be defined by $\varphi\begin{pmatrix} a & b \\ b & a \end{pmatrix} = a - b.$ Show that φ is a ring homomorphism and $R/\ker\varphi$ is isomorphic to $\mathbb{Z}$.

Examine whether ker φ is a prime ideal or maximal ideal. 1+2+1+1

(c) (i) Show that the ideal $S = \{(m, n) \in \mathbb{Z} \times \mathbb{Z} : 5|n\}$ of $Z \times Z$ is a maximal ideal.

(ii) Find the maximal ideals in Z_6. 3+2

(d) Define characteristic of a ring. If the characteristic of an integral domain D be a non-zero number p, then prove that the order of every non-identity element in the group $(D, +)$ is p. 1+4

(e) (i) Prove that every subring of the ring Z_n is an ideal.

(ii) Let $R = (2Z, +, \cdot)$ and $I = (6Z, +, \cdot)$. Find the elements of the quotient ring R/I. Is R/I a ring with unity? Justify your answer.

2+3

(f) Let $C[0, 1]$ be the ring of all real valued continuous functions on the closed interval $[0, 1]$. Let $\phi : C[0, 1] \to \mathbb{R}$ be defined by $\phi(f) = f\left(\frac{1}{2}\right)$, $f \in C[0,1]$. Show that ϕ is an onto ring homomorphism. Prove that $C[0,1]/\ker\phi$ is isomorphic to $\mathbb{R}$. 4+1

(g) (i) Let D be an integral domain. Prove that the ideal $A = \{(0, x) : x \in D\}$ is a prime ideal of $D \times D$. Here '0' is the zero element of D.

(ii) Prove that in a ring R with identity where $a^2 = a$ for all $a \in R$, every prime ideal is maximal. 2+3

(h) If R is a non-trivial ring such that $xR = R$ for every non-zero $x \in R$, then prove that R is a skew field. 5

Unit - II

3. Answer ***any four*** questions : 5×4=20

(a) Determine the linear mapping $T : R^3 \to R^3$ that maps the basis vectors (0, 1, 1), (1, 0, 1), (1, 1, 0) of R^3 to the vectors (2, 1, 1), (1, 2, 1), (1, 1, 2) respectively. Find ker T and ImT. Verify that dim ker$T$ + dim ImT = 3. 2+1+1+1

(b) Verify Cayley–Hamilton theorem for the matrix A. Express A^{-1} as a polynomial in A and then compute A^{-1} where $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 3 & 1 \end{pmatrix}$. 5

(c) (i) Find dim $(S \cap T)$ where S and T are subspace of the vector space R^4 given by :

$$S = \{(x, y, z, w) \in R^4 : 2x + y + 3z + w = 0\}$$
$$T = \{(x, y, z, w) \in R^4 : x + 2y + z + 3w = 0\}.$$

(ii) Find the eigenvalues and corresponding eigenvectors of $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$. 2+3

(d) Show that $U = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a + b = 0\right\}, W = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : c + d = 0\right\}$ are the subspaces of $\mathbb{R}_{2\times2}$. Find the dimension of U, $U \cap W$ & $U+W$. 2+1+1+1

(e) Prove that the set $S=\{(1,1,0),(1,0,1),(0,1,1)\}$ is a basis of the vector space $\mathbb{R}^3$. Show that the vector (1, 1, 1) may replace any one of the vectors of the set S to form a new basis. 2+3

(f) (i) Let $\phi : V \to W$ be an isomorphism where V and W are finite dimensional vector spaces over a field F. For any linearly independent subset S of V, prove that $\phi(S)$ is linearly independent in W.

(ii) A linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $T(x_1, x_2, x_3) = (3x_1 - 2x_2 + x_3,\ x_1 - 3x_2 - 2x_3)$ for all $(x_1, x_2, x_3) \in \mathbb{R}^3$. Find the matrix of T relative to the ordered bases (0, 1, 0), (1, 0, 0), (0, 0, 1) of $\mathbb{R}^3$ and (0, 1), (1, 0) of $\mathbb{R}^2$. 3+2

(g) Let A be a 3×3 real matrix having the eigenvalues 1, 2, 0. Let $\begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}$ be the eigenvectors of A corresponding to the eigenvalues 1, 2, 0 respectively. Find the matrix A. 5

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