2020

MATHEMATICS — HONOURS

Paper : DSE-A-3

(Bio Mathematics)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Group - A

(Marks : 20)

1. Answer the following multiple choice questions with only one correct option. Write the correct option with proper justification. (1+1)×10

(a) In an ecosystem following logistic growth model initial population was 900 with growth rate constant value 0.1. If the carrying capacity of the ecosystem is 1000, what is the instantaneous rate of change of population?

(i) 10 (ii) 25

(iii) 11.1 (iv) 9

(b) The steady state $x^* = 5$ for the equation $\frac{dx}{dt} = 3x\left(1 - \frac{x}{5}\right)$ is

(i) asymptotically stable (ii) stable but not asymptotically stable

(iii) unstable (iv) none of these.

(c) The Holling type-II function represents a

(i) straight line (ii) circle

(iii) parabola (iv) hyperbola.

(d) If r be the per capita growth rate and K, the environmental carrying capacity, then in the density dependent equation

$$\frac{dN}{dt} = rNf(N),$$

for environmental regulation, the function $f(N)$ should satisfy :

(i) $f(0) = 0,\ f(K) = 0,\ f(N) < 0$ when $N > K$

(ii) $f(0) = 1,\ f(K) = 0,\ f(N) < 0$ when $N > K$

(iii) $f(0) = 1,\ f(K) = 1,\ f(N) < 0$ when $N > K$

(iv) None of these.

(e) A system has the characteristic equation $\lambda^3 + 4K\lambda^2 + (5 + K)\lambda + 10 = 0$.

The range of K for a stable system is

(i) $K > 0.46$ (ii) $K < 0.46$

(iii) $0 < K < 0.46$ (iv) None of these.

(f) Consider the reaction diffusion model $u_t = \frac{u^2}{v} - bu + \nabla^2 u,\ \ v_t = u^2 - v^2 + \nabla^2 v$, b is a parameter.

The condition for stability of the positive spatially uniform steady state solution to spatially uniform perturbation is

(i) $b = 1$ (ii) $b > 1$

(iii) $0 < b < 1$ (iv) none of these.

(g) In the discrete model $N_{t+1} = (1+r)N_t - \frac{r}{K} N_t^2$, $r > 0, K > 0$ (constants), the non-trivial equilibrium is asymptotically stable for

(i) $0 < r < 1$ (ii) $r > 2$

(iii) $1 < r < 2$ (iv) None of these.

(h) For the predator-prey model $\begin{cases} \dfrac{dx}{dt} = rx\,(1-y) \\ \dfrac{dy}{dt} = my\,(1-x), \end{cases}$

(where m, r are positive constants), the equilibrium (0, 0) is

(i) saddle point (ii) stable focus

(iii) centre (iv) unstable focus.

(i) The transcritical bifurcation with μ as bifurcation parameter can be found in the system

(i) $\dot{x} = \mu - x^2$ (ii) $\dot{x} = \mu x - x^2$

(iii) $\dot{x} = \mu x - x^3$ (iv) $\dot{x} = \mu x + x^3$

(j) The steady state $x^* = 3$ of the difference equation $x_{n+1} = x_n e^{3 - x_n}$ is

(i) asymptotically stable (ii) stable but not asymptotically stable

(iii) unstable (iv) none of these.

Group - B

Unit - I

(Marks : 15)

Answer ***any one*** question.

2. (a) What are the defects of the Malthusian growth model? For a certain microorganism, birth is by budding of a fully formed copy of itself. Suppose that under reasonable favourable laboratory conditions, such birth occur on average four times per day, and an individual lives, on average, one day. Write the differential equation for the population, $p(t)$, of the organism as a function of time. Also find the solution, given at time $t = 0$, the population size is 1000.

(b) Define functional response in a prey-predator interaction. Describe different Holling type functional responses in mathematical form and draw the sketches of the response curves. Explain the merits or demerits, if any of different Holling type response functions. (2+3+3)+(1+3+2+1)

3. (a) From the flowchart of the SIR model without vertical transmission

$$\begin{tikzcd} {} \arrow[r, "bK"] & \boxed{S} \arrow[r, "\beta SI"] \arrow[d] & \boxed{I} \arrow[r, "\gamma I"] \arrow[d] & \boxed{R} \arrow[d] \\ & \mu S & \mu I & \mu R \end{tikzcd}$$

construct the SIR model and explain.

Show that if $\beta K < (\mu + \gamma)$ there is only one equilibrium which is locally asymptotically stable.

Also prove that if $\beta K > (\mu + \gamma)$ there is another equilibrium which is also locally asymptotically stable.

(b) Write short notes on the following :

(i) Allee effect

(ii) SIR epidemic model

(iii) Michaelis-Menten kinetics. (3+2+3)+(2+3+2)

Unit - II

(Marks : 20)

Answer ***any two*** questions.

4. (a) Find the steady states of the following prey-predator system :

$$\frac{dx}{dt} = x\left(1 - \frac{x}{13}\right) - \frac{xy}{x+10}, \quad \frac{dy}{dt} = y\left(\frac{x}{x+10} - \frac{3}{5}\right)$$

Investigate their stability.

(b) Write short note on (i) transcritical bifurcation (ii) Hopf bifurcation. (2+4)+(2+2)

5. What do you mean by limit cycle for a non-linear dynamical system?

Show that the following dynamical system possesses a limit cycle.

$$\left.\begin{aligned} \dot{x} &= y + x\,(1 - x^2 - y^2) \\ \dot{y} &= -x + y\,(1 - x^2 - y^2) \end{aligned}\right\}.$$

Also study nature of the trivial equilibrium point. 2+5+3

6. Consider the reaction-diffusion system given by

$$u_t = a - u + u^2 v + D_1 \nabla^2 u$$
$$v_t = b - u^2 v + D_2 \nabla^2 v$$

where symbols have their usual meaning.

Find the non-trivial spatially uniform steady state solution (u^*, v^*) of the system. Show that it is a cross-activator-inhibitor system near (u^*, v^*) under certain conditions on the parameters. Find the ranges r_1 and r_2 of u and v. If $r_1 << r_2$, give an approximation to the critical wavelength of the pattern formed. 2+4+2+2

7. Let the dimensionless activator-inhibitor system be described by

$$\left.\begin{aligned} \frac{du}{dt} &= a - u + u^2 v \\ \frac{dv}{dt} &= b - u^2 v \end{aligned}\right\}$$

where a, b are positive constants.

(a) Identify the activator and inhibitor of the system.

(b) What biological phenomena are indicated by the non-linear system?

(c) Sketch the null clines.

(d) Is it possible to have positive multi-steady states? 2+3+3+2

Unit - III

(Marks : 10)

Answer ***any one*** question.

8. Consider the Nicholson–Bailey model

$$N_{t+1} = r\,N_t\,e^{-aP_t}$$

$$P_{t+1} = C\,N_t\left(1 - e^{-aP_t}\right)$$

where the symbols have their usual meaning.

Find the equilibria of the system and discuss their stability. 3+7

9. (a) The population dynamics of a species is governed by the discrete model

$$N_{t+1} = N_t \ \exp\left\{ r\left(1 - \frac{N_t}{K}\right)\right\}$$

where r, K are positive constants.

(i) Determine the steady states and their stability nature.

(ii) Show that a period-doubling bifurcation occurs at $r = 2$.

(b) A drug is administered every six hours. Let D_n be the amount of drug in the blood system at n-th interval. The body eliminates a certain fraction p of the drug during each time interval. If the initial blood administered is D_0, find D_n and $\lim\limits_{n\to\infty} D_n$. (3+2)+5

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