

2020

PHYSICS — HONOURS

Paper : DSE-A-1

(Advanced Mathematical Methods Theory)

Full Marks : 65

The figures in the margin indicate full marks.

*Candidates are required to give their answers in their own words
as far as practicable.*

Answer **question nos. 1 & 2**, and **any four** questions from the rest (**Q.3 to Q.8**).

- 1.** Answer **any five** questions : 2×5
- (a) If P_1 and P_2 are two projection operators, then under what condition is $P_1 + P_2$ also behaves like a projection operator?
 - (b) Show that any two vectors $|V_1\rangle$ and $|V_2\rangle$ ($\neq 0$) that are orthogonal to each other are linearly independent.
 - (c) Use transformation property of cartesian tensors to establish that every contraction reduces the rank of a tensor by 2.
 - (d) Show that δ_j^i is an isotropic tensor.
 - (e) If $|q\rangle$ is any eigenstate of operator \hat{Q} such that $\hat{Q}|q\rangle = q|q\rangle$ and suppose that \hat{C} is another operator with $\hat{C}|q\rangle = |-q\rangle$, show that $\hat{C}\hat{Q} = -\hat{Q}\hat{C}$.
 - (f) The line element in metric form is given by $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. Use Quotient law to argue that $g_{\mu\nu}$ is a covariant tensor of rank two.
 - (g) Let e_1, e_2, e_3 be the generators of a three dimensional Lie algebra with the commutation relation $[e_1, e_2] = 0, [e_2, e_3] = e_1 + e_2; [e_3, e_1] = -e_1$. Find $[e_1, [e_2, e_3]] + [e_3, [e_1, e_2]]$.
- 2.** Answer **any three** questions :
- (a) Given the set of vectors $u_1 = (2, -1, 0), u_2 = (1, 0, -1), u_3 = (3, 7, -1)$. Use Gram-Schmidt orthogonalization procedure, with the standard Euclidean inner product, to find an orthonormal set. 5
 - (b) Prove the vector identity $(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$
using $\epsilon_{ijk} \epsilon_{pqk} = \delta_{ip} \delta_{jq} - \delta_{iq} \delta_{jp}$. 5

Please Turn Over

- (c) An antisymmetric tensor $F_{\mu\nu}$ satisfies $\partial_\mu F^{\mu\nu} = j^\nu$.
- Show that $\partial_\nu j^\nu = 0$.
 - Find out the number of independent components of $F_{\mu\nu}$ in 4 dimensions.
 - Show that for any mixed tensor T_ν^μ , T_μ^μ is a scalar. 2+1+2
- (d)
 - Show that all $(n \times n)$ unitary matrices form a group under multiplication.
 - Show that all such matrices with determinant 1 forms a subgroup. 3+2
- (e) Show that all integers including zero form a group under addition. Write down the identity element. Do they form a group under subtraction? Justify. 2+1+2

3. Show that the matrix $A = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$

- is Hermitian.
- Find out its eigenvalues.
- Find out eigenvectors and show that they are orthogonal.
- Denoting the eigenvectors $|1\rangle$ and $|2\rangle$, show that $|1\rangle\langle 1| + |2\rangle\langle 2| = \mathbb{I}$. 1+2+(3+1)+3

4. Starting with a vector space consisting of functions

$f_0(x) = 1, f_1(x) = x, f_2(x) = x^2, \dots, f_n(x) = x^n$ and defining the scalar product as

$$\langle f_m | f_n \rangle \equiv \int_{-1}^1 f_m(x) f_n(x) dx$$

- Show that these functions are not orthogonal to each other.
- Starting from $f_0(x)$ and $f_1(x)$, use Gram-Schmidt method to construct first three polynomials $P_0(x)$, $P_1(x)$ and $P_2(x)$.
- Normalize these polynomials such that $P_n(x=1) = 1$.
- Show that $P_2(x)$ is an eigenfunction of the operator $(1-x^2) \frac{d^2}{dx^2} - 2x \frac{d}{dx}$. Find the eigenvalue. 2+4+2+2

5. Consider two vectors (cartesian) \vec{A} and \vec{B} in 3D. For an anticlockwise rotation about the z -axis by an angle θ .

- How do the components of the vectors change?

(b) Show that for a norm-preserving rotation $\sum_i A_i B_i$ is a scalar, using transformation properties.

(c) Define :

$$C_1 \equiv A_2 B_3 - A_3 B_2$$

$$C_2 \equiv A_3 B_1 - A_1 B_3$$

$$C_3 \equiv A_1 B_2 - A_2 B_1$$

Show that under this rotation C_1, C_2, C_3 transform like the components of a vector.

(d) Identify \vec{C} in terms of \vec{A} and \vec{B} . 2+2+5+1

6. (a) Write down the Moment of Inertia tensor I_{ij} explaining each term.

(b) Show that I_{ij} is symmetric.

(c) Show that I_{ij} transforms as a 2nd rank tensor.

(d) Construct the inertia matrix for a system of three point masses of 1 unit, 2 units and 1 unit placed at $(1, 1, -2)$, $(-1, -1, 0)$ and $(1, 1, 2)$ respectively. 2+1+3+4

7. (a) Construct the group multiplication table for the set of elements $\{1, i, -1, -i\}$. Find out the identity element. Find the inverse of the element i and -1 .

(b) Consider $\{1, -1\}$. Show that they form a subgroup of the above group.

(c) Consider two matrices $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; $B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Show that the map $1 \rightarrow A$ and $-1 \rightarrow B$ is a homomorphism. (3+1+2)+2+2

8. (a) Write down the generators for $SU(2)$ group in fundamental representation.

(b) Find out the non-zero structure constants.

(c) Find out the adjoint representation of $SU(2)$ group. 2+3+5
