2020

MATHEMATICS — HONOURS

Paper : DSE-B-3

(Linear Programming and Game Theory)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

1. Answer ***all*** questions with proper explanation / justification (***one*** mark for correct answer and ***one*** mark for justification) : 2×10

(a) Suppose that the objective function of an L.P.P. assumes its optimal value at more than one extreme point. Then

(i) the convex combination of these extreme points will improve the value of the objective function.

(ii) the value of the objective function will be different for different convex combinations of these extreme points.

(iii) it indicates that the number of basic feasible solutions is degenerate.

(iv) every convex combination of these extreme points also gives the optimal value of the objective function.

(b) Consider two sets

$$X = \{(x_1, x_2) \mid x_1 + x_2 \le 2,\ 2x_1 + 2x_2 \ge 8,\ x_1,\ x_2 \ge 0\}$$

and $Y = \{y \mid |y| \le 2\}$.

Then

(i) X is a convex set, but Y is not a convex set.

(ii) X is not a convex set, but Y is a convex set.

(iii) both X and Y are convex sets.

(iv) none of X and Y are convex sets.

(c) Consider a game of size $m \times n$ with pay-off matrix $A = (a_{ij})_{m \times n}$. If a fixed number be added to each element of A, then

(i) the optimal strategies remain unchanged.

(ii) the value of the game remains unchanged.

(iii) the value of the game is decreased by that number.

(iv) both the optimal strategies and the value of the game remain unchanged.

(d) The optimal mixed strategies for the players A, B and the value of the game (ν) with pay-off matrix

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} -4 & 6 \\ 2 & -3 \end{bmatrix} \end{array}$$

will be

(i) $\left(\frac{1}{3}, \frac{2}{3}\right), \left(\frac{3}{5}, \frac{2}{5}\right)$ and $\nu = 0$.

(ii) $\left(\frac{14}{15}, \frac{1}{15}\right), \left(\frac{1}{3}, \frac{2}{3}\right)$ and $\nu = 0$.

(iii) $\left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{4}{5}, \frac{1}{5}\right)$ and $\nu = 0$.

(iv) (1, 0), (0, 1) and ν = 0.

(e) The maximum number of basic solutions for an $m \times n$ LPP is

(i) n (ii) m (iii) ${}^nC_m$ (iv) $m + n - 1$.

(f) Given the system of constraints :

$x_1 + 2x_2 + 3x_3 + 4x_4 = 7$

$2x_1 + x_2 + x_3 + 2x_4 = 3$

(i) (0, 2, 0, 1) is a basic solution,

(ii) (1, 1, 0, 0) is a basic solution,

(iii) (0, 2, 1, 0) is a basic solution,

(iv) (2, 0, 1, 4) is a basic solution.

(g) The optimal solution of the following LPP

Maximize $Z = x_1 + 5x_2$

subject to $3x_1 + 4x_2 \le 6$

$x_1 + 3x_2 \ge 3$

$x_1, x_2 \ge 0$

is given by

(i) $x_1 = \frac{6}{5}, x_2 = \frac{3}{5}$

(ii) $x_1 = 0, x_2 = 1$

(iii) $x_1 = 2, x_2 = 0$

(iv) $x_1 = 0, x_2 = \frac{3}{2}$.

(h) Given the following cost matrix of a transportation problem

	1	2	3	
1	2	2	3	10
2	4	1	2	15
3	1	3	1	40
	20	15	30	

the cost of transportation according to North-West corner rule is given by

(i) 100 (ii) 152 (iii) 125 (iv) 215

(i) For the following cost matrix

		Machine		
		M_1	M_2	M_3
	J_1	4	4	6
Job	J_2	10	7	11
	J_3	4	5	12

the minimum cost of assignment is

(i) 15 units

(ii) 17 units

(iii) 20 units

(iv) 22 units

(j) Given the following LPP

Maximize $Z = x_1 + 2x_2$

subject to $2x_1 + 3x_2 \geq 4$

$3x_1 + 4x_2 = 5$

$x_1 \geq 0$ and x_2 is unrestricted in sign the dual LPP is

(i) Minimize $W = -4y_1 + 5y_2$

subject to $-2y_1 + 3y_2 = 1$

$-3y_1 + 4y_2 \geq 2$

where $y_1 \geq 0$ and y_2 is unrestricted in sign

(ii) Minimize $W = 4y_1 - 5y_2$

subject to $-2y_1 + 3y_2 \geq 1$

$-3y_1 + 4y_2 \leq 2$

where $y_1 \geq 0$ and y_2 is unrestricted in sign

(iii) Minimize $W = -4y_1 + 5y_2$

subject to $-2y_1 + 3y_2 \geq 1$

$-3y_1 + 4y_2 = 2$

where $y_1 \geq 0$ and y_2 is unrestricted in sign

(iv) None of these.

Unit - 1

2. Answer ***any two*** questions :

(a) A city hospital has the following daily requirements of nurses in the ongoing COVID period at the minimal level :

Period	**Clock Time (24 hours a day)**	**Minimal number of nurses required**
1	6 a.m. – 10 a.m.	3
2	10 a.m. – 2 p.m.	8
3	2 p.m. – 6 p.m.	16
4	6 p.m. – 10 p.m.	9
5	10 p.m. – 2 a.m.	21
6	2 a.m. – 6 a.m.	7

Nurses report to the hospital at the beginning of each period and work for 8(eight) consecutive hours. The hospital wants to determine minimal number of nurses to be employed, so that sufficient number of nurses are available for each period. Formulate this as Linear Programming model by setting up appropriate constraints and objective function. 5

(b) $x_1 = 2$, $x_2 = 4$ and $x_3 = 1$ is a feasible solution to the system of equations :

$$2x_1 - x_2 + 2x_3 = 2,$$
$$x_1 + 4x_2 \qquad = 18.$$

Reduce the feasible solution to a basic feasible one. 5

(c) Prove that every extreme point of the convex set of all feasible solutions of the system $AX = b$, $X \geq 0$, corresponds to a basic feasible solution. 5

(d) (i) Prove that the set of all feasible solutions of a linear programming problem is a convex set.

(ii) Examine whether the set $S = \{(x_1, x_2) : x_1 x_2 \geq 1, x_1 \geq 0, x_2 \geq 0\}$ is convex or not. 3+2

Unit - 2

3. Answer ***any one*** question :

(a) (i) Apply simplex method to prove that the L.P.P.

Maximize $Z = 2x_2 + x_3$

subject to $x_1 + x_2 - 2x_3 \leq 7$,

$-3x_1 + x_2 + 2x_3 \leq 3, x_1, \; x_2, x_3 \geq 0$,

has an unbounded solution.

(ii) Use two phase simplex method to solve the L.P.P.

Minimize $Z = 4x_1 + x_2$

subject to $x_1 + 2x_2 \leq 3$,

$4x_1 + 3x_2 \geq 6$,

$3x_1 + x_2 = 3, \; x_1, \; x_2 \geq 0$. 4+6

(b) Mention two situations of degeneracy that may occur in solving an L.P.P.

Prove that a situation of degeneracy occurs in solving the L.P.P. given by

Maximize $Z = 22x_1 + 30x_2 + 25x_3$

subject to $2x_1 + x_2 \leq 100$,

$2x_1 + x_2 + x_3 \leq 100$,

$x_1 + 2x_2 + 2x_3 \leq 100, \; x_1, x_2, x_3 \geq 0$.

Mention clearly the method to resolve the degeneracy you are applying in this problem. 2+7+1

Unit - 3

4. Answer ***any one*** question :

(a) (i) Suppose that a constraint in a given L.P.P. (considered to be primal) is an equality. Prove that the corresponding dual variable is unrestricted in sign.

(ii) Write down the dual of the following L.P.P. :

Maximize $Z = 2x_1 + 3x_2 - 4x_3$

subject to $3x_1 + x_2 + x_3 \leq 2$

$-4x_1 + 3x_3 \geq 4$

$x_1 - 5x_2 + x_3 = 5$

$x_1 \geq 0$, $x_2 \geq 0$ and x_3 is unrestricted in sign. 6+4

(b) Use duality to find the optimal solution, if any, of the L.P.P. : 10

Maximize $Z = 2x_1 + x_2$

subject to $x_1 + 2x_2 \le 10,$

$x_1 + x_2 \le 6,$

$x_1 - x_2 \le 2,$

$x_1 - 2x_2 \le 1,\ x_1, x_2 \ge 0.$

Unit - 4

5. Answer ***any three*** questions :

(a) Prove that the number of basic variables in a transportation problem is at most $(m + n - 1)$, where 'm' is the number of origins and 'n' the number of destinations. 5

(b) Find whether the solution $x_{13} = 50$ units, $x_{14} = 20$ units, $x_{21} = 55$ units, $x_{31} = 30$ units, $x_{32} = 35$ units and $x_{34} = 25$ units is an optimal solution of the following transportation problem. If not then modify it to obtain the optimal solution.

					Available units
	6	1	9	3	70
	11	5	2	8	55
	10	12	4	7	90
Required units	85	35	50	45	

5

(c) Solve the following Travelling Salesman Problem so as to minimize the cost per cycle.

		To				
	City	1	2	3	4	5
	1	∞	10	25	25	10
	2	1	∞	10	15	2
From	3	8	9	∞	20	10
	4	14	10	24	∞	15
	5	10	8	25	27	∞

5

(d) Reduce the following pay-off matrix to a 2 × 2 matrix by dominance property and then solve the game problem, where A is the maximizing player and B is the minimizing player :

$$\begin{array}{cc} & \text{B} \\ \text{A} & \begin{bmatrix} 2 & 2 & 1 & -2 & -3 \\ 4 & 3 & 4 & -2 & 0 \\ 5 & 1 & 2 & 5 & 6 \\ 1 & 2 & 1 & -3 & -3 \end{bmatrix} \end{array}$$

5

(e) Find the optimal strategies for both persons and the value of the game for zero sum two person game whose pay-off matrix is as follows : 5

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \left|\begin{array}{rr} 1 & -3 \\ 3 & 5 \\ -1 & 6 \\ 4 & 1 \\ 2 & 2 \\ -5 & 0 \end{array}\right| \end{array}$$

(f) (i) State the fundamental theorem of rectangular game.

(ii) Write the standard form of L.P.P. corresponding to the following problem of game from the point of view of player B :

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{bmatrix} 1 & -1 & 3 \\ 3 & 5 & -3 \\ 6 & 2 & -2 \end{bmatrix} \end{array}$$

(iii) State how you would obtain the optimal solution for player A from the above formulation.

1+3+1