2018

MATHEMATICS – HONOURS

Paper : CC-1

Unit : 1, 2, 3

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Unit - 1

1. Answer ***all*** questions. Each question has four possible options, of which exactly one is correct. Write the correct option number against each question : 2×4

(a) The curve $y = e^{2x}$ is

(i) convex with respect to the y-axis

(ii) convex with respect to the x-axis

(iii) concave with respect to the y-axis

(iv) concave with respect to the x-axis

(b) The asymptote of the curve $y = x + \sin x$ is

(i) $y = x$ (ii) $y = -x$

(iii) $y = {}^1/_x$ (iv) does not exist

(c) A necessary condition for existence of the limit $\lim\limits_{x\to 0} \dfrac{a\sin 2x - b\sin x}{x^3}$ is

(i) $a = 2b$ (ii) $a = b$

(iii) $a + b = 0$ (iv) $2a = b$

(d) The area of the region bounded by $x = \pm 1$, $y = 0$ and $y = x^2$ is

(i) $\frac{1}{3}$ sq. unit (ii) $\frac{2}{3}$ sq. unit

(iii) 1 sq. unit (iv) 0 sq. unit

2. Answer ***any three*** questions : 4×3

(a) If $y = \dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$, $|x|<1$; show that

$(1 - x^2)y_{n+2} - (2n + 3)xy_{n+1} - (n + 1)^2 y_n = 0$

[Notations have usual meanings]

(b) Find the curvature of $x = a\sin 2t\ (1 + \cos 2t)$, $y = a\cos 2t\ (1 - \cos 2t)$ at 't'.

(c) Prove that the envelope of the family of circles drawn upon the radii vectors of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ as diameters is $\left(x^2+y^2\right)^2 = a^2x^2+b^2y^2$.

(d) If $I_{m,n} = \int \frac{\sin^m x}{\cos^n x}dx$, where m, n are both positive integers, prove that

$$I_{m,n} = \frac{1}{n-1}\,\frac{\sin^{m-1} x}{\sin^{n-1} x} - \frac{m-1}{n-1} I_{m-2,\,n-2}.$$

(e) Show that the length of the parabola $y^2 = 4ax$ cut off by its latus rectum is $2a\left[\sqrt{2}+\log(1+\sqrt{2})\right]$, where $a > 0$.

Unit - 2

3. Answer ***all*** questions. Each question has four possible options, of which exactly one is correct. Write the correct option number against each question : 2×2

(a) The polar equation $12 - 4r + \sqrt{2}\, r\cos\theta = 0$ represents

(i) a straight line (ii) an ellipse

(iii) a hyperbola (iv) a parabola

(b) The centre of the circle $x^2 + y^2 + z^2 - 2y - 4z = 11$, $x + 2y + 2z = 15$ is at

(i) (1, 3, 4) (ii) (2, 1, 1)

(iii) (5, 0, 3) (iv) (1, –1, 4)

4. Answer ***any five*** questions :

(a) Reduce the following equation to its canonical form : 5

$4x^2 - 4xy + y^2 + 2x - 26y + 9 = 0$

(b) Show that the straight line $r\cos(\theta - \alpha) = p$ touches the conic $\frac{l}{r} = 1 + e\cos\theta$, if $(l\cos\alpha - ep)^2 + l^2\sin^2\alpha = p^2$.

Also find the equation of the normal to the curve at the point of contact. 3+2

(c) Show that the equation to the plane containing the straight line $\frac{y}{b}+\frac{z}{c}=1$, $x = 0$ and parallel to the straight line $\frac{x}{a}-\frac{z}{c}=1$, $y = 0$ is $\frac{x}{a}-\frac{y}{b}-\frac{z}{c}+1=0$. If $2d$ is the shortest distance between the lines, then prove that $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{d^2}$. 3+2

(d) A variable plane has intercepts on the co-ordinate axes, the sum of whose squares is a constant K^2. Show that the locus of the foot of the perpendicular from the origin to the plane is
$(x^2 + y^2 + z^2)^2 (x^{-2} + y^{-2} + z^{-2}) = K^2$. 5

(e) Deduce a condition for coplanarity of two given straight lines in three dimensions. Also find the angle between two such lines. 3+2

(f) The section of the cone whose guiding curve is the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$, $z = 0$ by the plane $x = 0$ is a rectangular hyperbola. Show that the locus of the vertex is the surface $\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2}=1$. 5

(g) Find the equation of the cylinder whose guiding curve is the ellipse $4x^2 + y^2 = 1, z = 0$ and generators are parallel to the straight line $\frac{x}{2}=\frac{y}{-1}=\frac{z}{3}$. 5

(h) Find the equations of the generating lines of the hyperboloid $\frac{x^2}{4}+\frac{y^2}{9}-\frac{z^2}{16}=1$ passing through the point $(2, -1, \frac{4}{3})$. 5

Unit - 3

5. Answer ***all*** questions. Each question has four possible options, of which exactly one is correct. Write the correct option number against each question : 2×4

(a) The angle between the planes $\vec{r}.(2\hat{i}+3\hat{j}+\hat{k})=7$ and $\vec{r}.(3\hat{i}-2\hat{j}+5\hat{k})=5$ is

(i) $\frac{\pi}{2}$ (ii) $\cos^{-1}\left(\frac{5}{\sqrt{14}\sqrt{38}}\right)$

(iii) $\cos^{-1}\left(\frac{17}{\sqrt{14}\sqrt{38}}\right)$ (iv) $\sin^{-1}\left(\frac{-5}{\sqrt{38}}\right)$

(b) The distance of the point (1, 2, 3) from the plane $\vec{r}.(3\hat{i}-4\hat{j}+\hat{k})=-1$ is

(i) $\sqrt{26}$ units (ii) $\frac{1}{\sqrt{26}}$ unit

(iii) 14 units (iv) 0 unit

(c) If $\vec{r}(t)$ is a differentiable vector function of a scalar variable t such that

$$\vec{r}(t)=\begin{cases} 2\hat{i}-\hat{j}+2\hat{k} & \text{when } t=2 \\ 4\hat{i}-2\hat{j}+3\hat{k} & \text{when } t=3 \end{cases}$$

then the value of the integral $\int_2^3\left(\vec{r}.\frac{d\vec{r}}{dt}\right)dt$ is

(i) 10 (ii) 100

(iii) 16 (iv) $\frac{1}{10}$

(d) If $\vec{r}=(\sin t)\hat{i}+(\cos t)\hat{j}+(2t)\hat{k}$, then the value of $\left|\frac{d^2\vec{r}}{dt^2}\right|$ is

(i) 1 (ii) 2

(iii) 3 (iv) 4

6. Answer ***any two*** questions : 4×2

(a) Prove that $\left[\vec{a}\times\vec{b}\ \ \vec{c}\times\vec{d}\ \ \vec{e}\times\vec{f}\right]=\left[\vec{a}\ \vec{b}\ \vec{e}\right]\left[\vec{c}\ \vec{d}\ \vec{f}\right]-\left[\vec{a}\ \vec{b}\ \vec{f}\right]\left[\vec{c}\ \vec{d}\ \vec{e}\right]$.

(b) Use vector method to find the point of intersection of the straight line joining the points (8, –3, 5) and (2, –1, 1) and the plane passing through the points (3, 0, 1), (4, –1, 2) and (2, 1, –3).

(c) Find the vector equation of the plane passing through the point $\left(8\hat{i}+2\hat{j}-3\hat{k}\right)$ and perpendicular to each of the planes $\vec{r}.\left(2\hat{i}-\hat{j}+2\hat{k}\right)=0$ and $\vec{r}.\left(\hat{i}+3\hat{j}-5\hat{k}\right)+5=0$.

(d) If $\vec{r}(t)=5t^2\hat{i}+t\hat{j}-t^3\hat{k}$, evaluate $\int_1^2\left(\vec{r}\times\frac{d^2\vec{r}}{dt^2}\right)dt$.