2023

PHYSICS — HONOURS

Paper : CC-5

Full Marks : 50

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Answer ***question no.* 1** and ***any four*** questions from the rest.

1. Answer ***any five*** questions : 2×5

(a) Show that $\int_{-\pi}^{\pi} \cos mx \sin nx \, dx = 0$ for all n and m.

(b) Find the constant term in the expansion for $\cos^2 x$ as a Fourier series in the interval $(-\pi, \pi)$.

(c) Using generating function for Legendre polynomial, show that $P_n(x) = (-1)^n P_n(-x)$.

(d) Solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$, given that $u(0, t) = 4e^{-3t}$.

(e) Prove that $\Gamma(n) = n!$, where n is a positive integer.

(f) Show that for Beta function, $\beta(m, n) = \beta(n, m)$.

(g) For a Binomial distribution $f(r) = {}^nc_r \, p^r q^{n-r}$, where $p + q = 1$, find $\langle r \rangle$.

2. (a) A periodic function $f(x)$ with period 2π is defined as $f(x) = x^2$, $-\pi \leq x \leq \pi$.

(b) Expand $f(x)$ in a Fourier series and hence show that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

(c) From the above Fourier expansion, obtain the Fourier expansion of the periodic function

$$g(x) = x, \text{ for } -\pi < x < \pi.$$

Discuss the convergence of the series at $x = 0$ and $x = \pi$ and hence find the value of the series at $x = 0$ and $x = \pi$. 4+2+(2+2)

3. (a) Consider $f(x) = \begin{cases} f_0, & -a \le x \le a \\ 0, & \text{otherwise} \end{cases}$, where f_0 is a constant.

(b) Find the value of $g(k)$ at $k = 0$, hence plot $f(x)$ and $g(k)$.

(c) Define width of $g(k)$ as the distance between the first zeros of $g(k)$ on either side of $k = 0$. If width of $f(x)$ is $2a$ then show that the product of widths of $g(k)$ and $f(x)$ is constant.

5+(1+3)+1

4. (a) Consider the differential equation $x(x + 1)y'' + (3x + 1)y' + y = 0$

(i) Check whether $x = 0$ is an ordinary point or a regular singular point.

(ii) Find the indicial equation.

(iii) From the indicial equation, find two linearly independent solutions of the given differential equation.

(b) Consider the differential equation $\dfrac{1}{\sin\theta}\dfrac{d}{d\theta}\left(\sin\theta\dfrac{dT}{d\theta}\right) + l(l+1)T = 0$ for $T(\theta)$.

Show that the substitution $w = \cos\theta$ converts it into Legendre's equation for $P(w) = T(\theta)$.

(2+2+3)+3

5. (a) The generating function for the Hermite polynomials $H_n(x)$ satisfies

$$e^{2xt - t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}.$$

Use this relationship to prove that $H_n'(x) = 2nH_{n-1}(x)$.

(b) Using Rodrigue's formula for Legendre Polynomials

$$P_n(x) = \frac{1}{n!2^n}\frac{d^n}{dx^n}(x^2 - 1)^n,$$

show that $P_n(1) = 1$.

(c) Calculate the mean and variance of random variable x whose probability density function $p(x)$ is given as

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in R$$

2+3+(2+3)

6. (a) Find the value of $\Gamma\left(\frac{1}{2}\right)$.

(b) Write the integral $\int_0^1 \frac{x^3}{\sqrt{1-x}}\,dx$ in the form of a Beta function and hence evaluate it.

[Use $\beta(m, n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta$]

(c) Prove that $\beta(m, n) = \frac{\Gamma(m)\ \Gamma(n)}{\Gamma(m+n)}$. 3+(1+2)+4

7. (a) Find a solution $U(x, t)$ of the boundary-value problem $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$, if $U(x, 0) = \sin \pi x$.

(b) Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ in the region $0 \le x \le a$, $0 \le y < \infty$ satisfying the conditions

$u(0, y) = 0, \quad u(a, y) = 0$

$u(x, 0) = A\left(1 - \frac{x}{a}\right)$ and $u(x, \infty) = 0$ 4+6