2024

MATHEMATICS — HONOURS

Paper : CC-12

(Unit - I + II)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

$\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ respectively denote the set of all integers, set of all real numbers and set of all complex numbers.

1. Choose the correct answer with proper justification (1 mark for right answer and 1 mark for justification) : 2×10

(a) Which of the following is an Abelian group?

(i) $S_3 \times S_3$ (ii) $A_3 \times S_3$

(iii) $\mathbb{Z}_3 \times S_3$ (iv) $A_3 \times A_3$.

(b) Consider the group $\mathbb{Z}_8$. Then the group $Aut(\mathbb{Z}_8)$ of automorphism of $\mathbb{Z}_8$ is isomorphic to

(i) $\mathbb{Z}_2$ (ii) $\mathbb{Z}_4$

(iii) $\mathbb{Z}_6$ (iv) $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(c) Up to isomorphism, the number of Abelian groups of order 108 is

(i) 9 (ii) 12

(iii) 5 (iv) 6.

(d) If $|\text{Aut } G| > 1$, then which of the following is true?

(i) $|G| > 1$ (ii) $|G| \leq 2$

(iii) $|G| > 2$ (iv) None of the above is true.

(e) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear mapping defined by $T((a, b)) = (2a + b, a - 3b)$ and T^* be the adjoint of T, then $T^*((1, 1))$ is equal to

(i) $(3, -2)$ (ii) $(-1, -2)$

(iii) $(3, 2)$ (iv) $(-1, 2)$.

(f) The quadratic form $ax^2 + bxy + cy^2$ is positive definite if and only if

(i) $a > 0$ (ii) $4ac > b^2$

(iii) $a > 0$ and $4ac > b^2$ (iv) $a > 0$ and $4ac < b^2$.

(g) Let $V(F)$ be an inner product space and $T: V \to V$ be a linear operator on V. If T^* is the adjoint operator of T and $a(\neq 0) \in F$, then

(i) $(aT)^* = aT^*$ (ii) $(aT)^* = a^{-1}T^*$

(iii) $(aT)^* = -aT^*$ (iv) $(aT)^* = \bar{a}T^*$

where $\bar{a}$ is the conjugate of a.

(h) If V is a finite dimensional vector space over the field F and V^0 is the annihilator of V, then

(i) $V^0 = V^*$ (ii) $V^0 = \{O\}$

(iii) $V^0 = \emptyset$ (iv) $V^0 \supset V^*$.

where V^* is the dual space of V and O is the zero functional on V.

(i) Let P be the subspace of $\mathbb{R}^3$ generated by (1, 1, 0) and (0, 1, 1). The orthogonal complement of P is a subspace generatd by the vector

(i) (1, 1, 1) (ii) (–1, 1, –1)

(iii) (1, 1, –1) (iv) (–1, –1, 1).

(j) If A and B are two subspaces of an inner product space $V(F)$ such that $A \subset B$, then which of the following is true?

(i) $A^\perp \subset B^\perp$ (ii) $B^\perp \subset A^\perp$

(iii) $A^\perp = B^\perp$ (iv) None of these.

Where $A^\perp$ and $B^\perp$ are the orthogonal complement of A and B respectively.

Unit - I

(Group Theory)

2. Answer *any four* questions :

(a) (i) Let H, K be two groups. Prove that $H \times K$ is commutative if and only if both H, K are commutative.

(ii) Give an example of a non-commutative group of order 66. 4+1

(b) (i) If $Z(G)$ be the centre of a group G, then prove that $\frac{G}{Z(G)} \simeq Inn(G)$.

(ii) Prove that $\mathbb{R}^* \simeq \mathbb{R}^+ \times \mathbb{Z}_2$, where $\mathbb{R}^*$ is the set of all non-zero reals and $\mathbb{R}^+$ is the set of all positive reals. 3+2

(c) (i) State fundamental theorem of finite Abelian groups.

(ii) Find all Abelian groups of order 720. 2+3

(d) (i) Let G be a group. Show that the mapping $f: G \to G$ defined by $f(a) = a^{-1}$ for all $a \in G$ is an automorphism if and only if G is an Abelian group.

(ii) Prove or disprove : For two groups H, K if $Aut(H) \simeq Aut(K)$, then $H \simeq K$. 3+2

(e) Let $f: G \to G$ be a homomorphism. If f commutes with every inner automorphism of G, then prove that

(i) $K = \{x \in G;\ f^2(x) = f(x)\}$ is a normal subgroup of G.

(ii) G/K is Abelian. 3+2

(f) Prove that every non-cyclic group of order p^2, p is a prime number, is isomorphic to external direct product of two cyclic groups each of order p. 5

(g) Suppose G is a finite Abelian group and p is a divisor of order of G, where p is a prime number. Prove that there is an element $a(\neq e)$ in G such that $a^p = e$, where e is the identity element in G. 5

Unit - II

(Linear Algebra)

3. Answer ***any five*** questions :

(a) Diagonalise the matrix $\begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 3 \end{pmatrix}$. 5

(b) Reduce the quadratic form $7x^2 - 2xy + 7y^2 - 16xz + 16yz - 8z^2$ to its canonical form. 5

(c) Find the Jordan normal form of the matrix $A = \begin{pmatrix} 3 & 1 & -2 \\ -1 & 0 & 5 \\ -1 & -1 & 4 \end{pmatrix}$. 5

(d) Use Gram-Schmidt orthonormalization process to find an orthonormal basis of $\mathbb{R}^3$ from the basis $\{(1, 0, 1), (1, 1, 1), (1, 3, 4)\}$. 5

(e) Define invariant subspace of a linear operator.

Consider the linear transformation $T: M_2(\mathbb{R}) \to M_2(\mathbb{R})$ defined by $T((A)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A$ for all $A \in M_2(\mathbb{R})$. Examine whether the subspace $W = \{A \in M_2(\mathbb{R}) : A^t = A\}$ is T-invariant. 1+4

(f) (i) Prove that $\|x + y\| \leq \|x\| + \|y\|$, where x, y be any two elements in an inner product space V.

(ii) If x and y are two vectors in a complex inner product space V, then prove that

$4(x, y) = \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2$,

where (x, y) denotes the inner product of x and y in $V(\mathbb{C})$. 2+3

Please Turn Over

(g) (i) Find the Hessian matrix of the function $f(x, y) = x^3 - 2xy - y^6$ at $(1, 2)$.

(ii) Let V be an n-dimensional vector space over the field F. Find the minimal polynomial of the identity operator $I_V : V \to V$. 2+3

(h) If T_1 and T_2 are two linear operators in a finite dimensional inner product space, then prove that

(i) $(T_1 + T_2)^* = T_1^* + T_2^*$

(ii) $(T_1 T_2)^* = T_2^* T_1^*$,

where T_1^* and T_2^* are the adjoint operators of T_1 and T_2 respectively. 3+2
