2024

MATHEMATICS — HONOURS

Paper : DSE-A-1.1 , DSE-A-1.2 and DSE-A-1.3

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations have usual meanings.

Paper : DSE-A-1.1

(Advanced Algebra)

Full Marks : 65

Group - A

(Marks : 20)

1. Answer *all* questions. In each question **one** mark is reserved for selecting the correct option and **one** mark is reserved for justification : (1+1)×10

(a) Which of the following statements is correct?

(i) A group of order 22 is simple (ii) A group of order 26 is simple

(iii) A group of order 34 is simple (iv) A group of order 46 is not simple.

(b) Let G be a group of order 15. Then which of the following is true?

(i) G is cyclic (ii) G is not isomorphic to $\mathbb{Z}_{15}$

(iii) G is simple (iv) None of these.

(c) Let G be a group of order 77. Then which of the following is true?

(i) G contains 11 Sylow 7-subgroups of order 7

(ii) G contains 7 Sylow 11-subgroups of order 11

(iii) G is simple

(iv) G contains a unique Sylow 7-subgroups of order 7.

(d) Which of the following are all the associates of [6·] in $\mathbb{Z}_{10}$?

(i) [2], [4], [6], [9] (ii) [2], [4], [7], [9]

(iii) [3], [5], [7], [8] (iv) [2], [4], [6], [8].

(e) Which of the following is true?

(i) A regular ring is a field

(ii) A field is a regular ring

(iii) An integral domain is a regular ring

(iv) A regular ring is a Boolean ring.

(f) Identify the correct statement from the following :

(i) $\mathbb{Z}[x]$ is a principal ideal domain (ii) If F is a field, then $F[x]$ is a principal ideal domain.

(iii) $\mathbb{Z}$ is not a principal ideal domain (iv) $\mathbb{Z}[i]$ is not an Euclidean domain.

(g) The number of Sylow 3-subgroups of S_4 is

(i) 2 (ii) 3

(iii) 4 (iv) 5.

(h) For any group G, which of the following can be the order of the group $\frac{G}{Z(G)}$?

(i) 15 (ii) 19

(iii) 55 (iv) 77.

(i) In the ring $\mathbb{Z}_{12}$, which of the following statements is true?

(i) [3] is a prime element (ii) [3] is an irreducible element

(iii) [5] is an irreducible element (iv) [5] is a prime element.

(j) Which of the following is not a regular ring?

(i) $\mathbb{Z}$ (ii) $\frac{\mathbb{Z}}{7\mathbb{Z}}$

(iii) $\mathbb{Q}$ (iv) $\mathbb{R}$.

Group - B

(Marks : 15)

2. Answer *any three* questions :

(a) (i) Consider the alternating group A_3 on the set $S = \{1, 2, 3\}$. Prove that there exists a group action of A_3 on S.

(ii) Prove that every group of order p^2 (where p is a prime) is commutative. 3+2

(b) (i) If G is a group of order p^n where p is a prime and n is a positive integer, then show that the centre $Z(G) \neq \{e\}$, where 'e' is the identity element of G.

(ii) Prove or disprove : There are 6 elements of order 7 in a group of order 28. 3+2

(c) Let G be a finite commutative group of order n. If m is a positive integer such that m divides n, then prove that G has a subgroup of order m. 5

(d) What do you mean by conjugacy class equation? Write down conjugacy class equation of S_3 in detail. 2+3

(e) State and prove Sylow's second theorem. 5

Group - C

(Marks : 30)

3. Answer *any six* questions :

(a) Prove that $(\mathbb{Z}, +, \cdot)$ is a principal ideal domain. 5

(b) Define an Euclidean domain. Prove that every Euclidean domain is a principal ideal domain. 1+4

(c) (i) Justify the following statement by citing an example :
'An integral domain may not be a regular ring'.

(ii) Prove that every division ring is a regular ring. 3+2

(d) Prove that the polynomial $x^{p-1} + x^{p-2} + ... + x + 1$ is irreducible in $\mathbb{Z}[x]$, where p is a prime number. 5

(e) In a principal ideal domain R, prove that *l.c.m.* of any two non-zero elements $a, b \in R$ exists. 5

(f) State true or false : 'Every Euclidean domain is a unique factorization domain'. Justify your answer. 1+4

(g) Show that in the integral domain $\mathbb{Z}\left[i\sqrt{5}\right], 2+i\sqrt{5}$ is an irreducible element but not a prime element. 5

(h) (i) Let R be an integral domain and p be a prime element in R. Show that p is irreducible.

(ii) Show that [2] is a prime element but not an irreducible element in $\mathbb{Z}_{10}$. 3+2

(i) Find the field of quotient of the integral domain $\mathbb{Z}[i]$. 5

(j) Let R be a ring with identity. Prove that R is a regular ring if and only if every principal left ideal of R is generated by an idempotent element of R. 5

Paper : DSE-A-1.2

(Bio-Mathematics)

Full Marks : 65

Group - A

(Marks : 20)

1. Answer the following multiple choice questions with only one correct option. Choose the correct option with proper justification. (1+1)×10

(a) If a population has exponential growth $\frac{dN}{dt} = rN$ with growth rate $r(>0)$, the population will be double in time

(i) $\frac{1}{r}\log_e 2$ (ii) $\log_e 2$

(iii) $r\log_e 2$ (iv) $\frac{1}{r^2}\log_e 2$

(b) For the logistic model $\frac{dx}{dt} = rx\left(1-\frac{x}{K}\right)$, the maximum rate of growth of population size x is

(i) rK (ii) $\frac{rK}{2}$

(iii) $\frac{rK}{4}$ (iv) $\frac{rK}{6}$.

(c) For the harvesting model $\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right) - h$, where $r, K, h > 0$, the unique equilibrium exists if

(i) $h < \frac{rK}{4}$ (ii) $h = \frac{rK}{4}$

(iii) $h > \frac{rK}{4}$ (iv) $K = \frac{rh}{4}$.

(d) The one-dimensional system $\frac{dx}{dt} = \mu x - x^3$, where $\mu \in \mathbb{R}$ is a parameter, has a

(i) saddle node bifurcation (ii) pitchfork bifurcation

(iii) transcritical bifurcation (iv) No bifurcation.

(e) The non-trivial steady state of the system

$$\frac{dx}{dt}=x\left(1-\frac{2x}{5}\right)-xy,$$
$$\frac{dy}{dt}=(2x-1)y,$$

is

(i) $\left(\frac{1}{2},\frac{1}{5}\right)$ (ii) $\left(\frac{1}{2},1\right)$

(iii) $\left(\frac{1}{2},\frac{4}{5}\right)$ (iv) $\left(\frac{1}{2},\frac{2}{5}\right)$.

(f) For the system

$$\frac{dx}{dt}=2x-3y,$$
$$\frac{dy}{dt}=x+y,$$

the steady state (0, 0) is

(i) stable spiral (ii) saddle point

(iii) centre (iv) unstable spiral.

(g) The steady state (0, 0) of the two-dimensional system

$$\frac{dx}{dt}=x(100-4x-2y),$$
$$\frac{dy}{dt}=y(30-x-y),$$

is

(i) stable node (ii) stable focus

(iii) unstable node (iv) saddle.

(h) The Holling type-II functional response is

(i) a straight line (ii) a closed curve

(iii) a hyperbola (iv) a sigmoidal curve.

(i) The non-trivial steady state of the discrete model $x_{n+1}=x_n+rx_n\left(1-\frac{x_n}{k}\right)$; $r, k>0$, is asymptotically stable for

(i) $0<r<2$ (ii) $2<r<3$

(iii) $2<r<4$ (iv) None of these.

(j) The positive steady state of the difference equation $x_{n+1} = \dfrac{1}{2+x_n}$ is

(i) $2-\sqrt{2}$ (ii) $-1+\sqrt{2}$

(iii) $1+\sqrt{2}$ (iv) $2\sqrt{2}-1$.

Group - B

(Unit - I)

(Marks : 15)

Answer ***any one*** question.

2. (a) What is an equilibrium point or steady state of $\dfrac{dx}{dt} = f(x)$?

Suppose x^* is an equilibrium point of the system $\dfrac{dx}{dt} = f(x)$,

where $f(x)$ is a continuously differentiable function with $f'(x^*) \neq 0$.

Prove that x^* is asymptotically stable if $f'(x^*) < 0$ and unstable if $f'(x^*) > 0$.

(b) Find the analytical solution of the logistic equation :

$$\frac{dx}{dt} = rx\left(1-\frac{x}{K}\right),\ x(0) = x_0 > 0.$$

Hence find the limiting value of x as $t \to \infty$.

(c) Reduce the above logistic equation to dimensionless form by the substitution

$$u = \frac{x}{K} \text{ and } \tau = rt.$$

Find the steady states of the dimensionless system and discuss their stability.

(1+5)+(3+1)+(2+1+2)

3. (a) State all the basic assumptions of the Malthusian growth model $\frac{dN}{dt} = rN$. Solve this model analytically. Draw the Malthusian growth curves for different values of r. What are the defects of this model?

(b) Consider the following harvesting model :

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - EN,$$

where r, K, E are positive parameters and $E < r$.

Find the steady states and discuss their stability.

(c) Write a short note on transcritical bifurcation. (2+2+2+2)+(1+2)+4

Unit - II

(Marks : 20)

Answer *any two* questions.

4. Consider the following modified Lotka-Volterra model :

$$\frac{dX}{dT} = rX\left(1 - \frac{X}{K}\right) - bXY,$$

$$\frac{dY}{dT} = -dY + cXY,$$

where r, K, b, c, d are positive parameters.

(a) Using the substitutions $x = \frac{X}{K}, y = \frac{b}{r}Y$ and $t = rT$, reduce the above system in the following dimensionless form :

$$\frac{dx}{dt} = x(1 - x - y),$$

$$\frac{dy}{dt} = \beta y(x - \alpha),$$

where α, β are the new parameters to be determined.

(b) Find all the steady states of the dimensionless system and discuss their stability. 3+(3+4)

5. (a) Consider the following Lotka-Volterra competition model :

$$\frac{dx}{dt} = x(1 - x - \alpha y),$$

$$\frac{dy}{dt} = \rho y(1 - y - \beta x),$$

where α, β, ρ are positive parameters. Find the steady states. Show that the coexistence steady state (x^*, y^*) is asymptotically stable if $\alpha < 1$, $\beta < 1$.

(b) Using Dulac criterion with Dulac function $B(x, y) = \frac{1}{xy}$, show that the above system has no periodic orbit in the interior of the first quadrant. (2+3)+5

6. Consider the non-linear system

$$\frac{dx}{dt} = x\left\{2\left(1-\frac{x}{K}\right)-\frac{3y}{1+x}\right\},$$
$$\frac{dy}{dt} = y\left\{-\frac{1}{2}+\frac{x}{1+x}\right\}, K>1.$$

Find the steady states and discuss their stability. 10

7. (a) Consider the compartmental model

$$\frac{dS}{dt} = -\lambda SI + \alpha R,$$
$$\frac{dI}{dt} = \lambda SI - \mu I,$$
$$\frac{dR}{dt} = \mu I - \alpha R,$$

where the parameters λ, μ, α are positive and $S(0)$, $I(0)$, $R(0)$ are initial values of the susceptible, infected and recovered populations respectively.

(i) Show that at any time t, the total number of population is constant.

(ii) Reduce the system into a two-dimensional system involving S and I population only.

(iii) Discuss the stability of the coexistence steady state of the reduced system.

(b) Show that the following system :

$$\frac{dx}{dt} = x + y - x(x^2 + y^2),$$
$$\frac{dy}{dt} = -x + y - y(x^2 + y^2),$$

has a stable limit cycle. (1+1+3)+5

Unit - III

(Marks : 10)

Answer ***any one*** question.

8. (a) Draw the Cobweb diagram of the difference equation :

$$x_{n+1} = \frac{1}{2}x_n + 20 .$$

[Show at least three iterations.]

(b) Let (x^*, y^*) be a steady state of the two-dimensional discrete-time system :

$$x_{n+1} = f(x_n, y_n),$$
$$y_{n+1} = g(x_n, y_n).$$

Linearise the system about (x^*, y^*) and hence state the conditions for stability of (x^*, y^*). 4+6

9. (a) Find the steady states of the following difference equation :

$$x_{n+1} = x_n e^{r(1-x_n)}, \text{ (r being a positive parameter)},$$

and discuss their stability.

(b) Find the steady states of the following Nicholson-Bailey model :

$$H_{n+1} = bH_n e^{-aP_n}$$
$$P_{n+1} = cH_n\left(1 - e^{-aP_n}\right),$$

where the symbols have their usual meanings. Hence show that the coexistence steady state (x^*, y^*) is always unstable. 4+(2+4)

Paper : DSE-A-1.3

(Industrial Mathematics)

Full Marks : 65

1. Choose the correct answer with proper justification/explanation for each of the following multiple choice question (**one** mark for each correct answer and **one** mark for justification.) : 2×10

(a) The rank of the matrix $\begin{pmatrix} 1 & -2 & 3 & 4 \\ 2 & 3 & 6 & -9 \\ 9 & 3 & 27 & -15 \end{pmatrix}$ is

(i) 1 (ii) 2

(iii) 4 (iv) 3.

(b) The standard form of the line $\mathcal{L}_{1/2,\pi/6}(-\infty < s < \infty), -\infty < s < \infty$ is

(i) $x = \frac{\sqrt{3}}{4} + \frac{s}{2}, y = \frac{1}{4} + \frac{\sqrt{3}}{2}s$ (ii) $x = \frac{\sqrt{3}}{4} + \frac{s}{2}, y = \frac{1}{4} - \frac{\sqrt{3}}{2}s$

(iii) $x = \frac{\sqrt{3}}{4} - \frac{s}{2}, y = \frac{1}{4} - \frac{\sqrt{3}}{2}s$ (iv) $x = \frac{\sqrt{3}}{4} - \frac{s}{2}, y = \frac{1}{4} + \frac{\sqrt{3}}{2}s$.

(c) The degree of the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} = \left(1 + y + \frac{dy}{dx}\right)^{1/3}$ is

(i) 1 (ii) 1/3

(iii) 2 (iv) 3.

(d) The value of the integral $\int_{-\infty}^{\infty} xe^{-x^2} dx$ is

(i) $\frac{1}{2}$ (ii) 0

(iii) 1 (iv) $\sqrt{\pi}$.

(e) If $(x, y) \in \mathcal{L}_{t,\theta} = \{(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta) : -\infty < s < \infty\}$, then $x^2 + y^2$ is equal to

(i) $t^2 + s^2$ (ii) $t^2 + \theta^2$

(iii) $\theta^2 + s^2$ (iv) $4st$.

(f) The attenuation coefficeint of an X-ray beam measures

(i) proportion of the photons absorbed by each millimeter of a substance when an X-ray passes through it.

(ii) wavelength of the X-ray.

(iii) proportion of the photons which are not absorbed by a substance when an X-ray passes through it.

(iv) None of the above.

(g) If $\mathcal{R}f(t, \theta)$ denotes the Radon transform of f, which one of the following is true?

(i) $\mathcal{R}(\alpha f + \beta g) = \alpha^2 \mathcal{R}f + \beta^2 \mathcal{R}g$

(ii) $\mathcal{R}(\alpha f + \beta g) = \alpha \mathcal{R}f + \beta \mathcal{R}g$

(iii) $\mathcal{R}(\alpha f + \beta g) = (\alpha - 1)\mathcal{R}f + (\beta - 1)\mathcal{R}g$

(iv) $\mathcal{R}(\alpha f + \beta g) = \mathcal{R}f + \mathcal{R}g$.

(h) If $f(x) = e^{-2x^2}$, then Fourier Transform of f is

(i) $\sqrt{\pi}e^{-\frac{\omega^2}{8}}$ (ii) $\sqrt{\frac{\pi}{2}}e^{-\frac{\omega^2}{8}}$

(iii) $\sqrt{\pi}e^{-\frac{\omega^2}{4}}$ (iv) $\sqrt{\frac{\pi}{2}}e^{-\frac{\omega^2}{4}}$.

(i) Algebraic Reconstruction Techniques (ARTs) are techniques for reconstructing images

(i) that have no direct connection to the Radon inversion formula.

(ii) that are same as the Radon inversion formula.

(iii) that are connected to but not same as the Radon inversion formula.

(iv) None of the above.

(j) Polar form of the complex number $z = -1 + i$ is

(i) $\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ (ii) $\sqrt{2}(\cos 0 + i\sin 0)$

(iii) $\sqrt{3}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ (iv) $\sqrt{3}(\cos 0 + i\sin 0)$.

Unit - I

2. Answer *any two* questions :

(a) What do you mean by X-ray Computerized Tomography (CT)? Explain with example. 4+1

(b) Solve the differential equation $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=x^2-1+e^{3x}$. 5

(c) Find A^{-1} if $A^2=\begin{pmatrix} 14 & -1 & 2 \\ 7 & 21 & 6 \\ 60 & 6 & 49 \end{pmatrix}$. 5

(d) If $2\cos\theta = x+\frac{1}{x}$ and θ is real, prove that $2\cos n\theta = x^n+\frac{1}{x^n}$, n being an integer. 5

Unit - II

3. Answer *any two* questions :

(a) If $f: A \to B$ is a function whose inverse function $f^{-1}: B \to A$ exists and P, Q are two non-empty subsets of A, then prove that $f(P \cup Q) = f(P) \cup f(Q)$ and $f^{-1}(P \cup Q) = f^{-1}(P) \cup f^{-1}(Q)$. 2+3

(b) Consider a bar of unit length having non-homogeneous density distribution and suppose that the linear mass density of the bar is a given continuous function f. For each segment $[0, x]$ of the bar, the centroid function of the segment is given by

$$C(x)=\frac{\int_0^x uf(u)du}{\int_0^x f(u)du},$$

where it is given that $0 < C(x) < x$ for $x > 0$, $\lim_{x\to 0+} C(x) = 0$ and $C'(x) > 0\ \forall x \in (0.1)$. Using the above relations solve the inverse problem to construct a density function $f(x)$ that gives rise to $C(x)$. 5

(c) Suppose $\tilde{b}=[1,0,2]^t$, $A=\begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 1 & 1 \end{pmatrix}$ and $\tilde{x}=[x_1, x_2]^t$. Show that the system $A\tilde{x}=\tilde{b}$ has no ordinary solution but that it has a unique least square solution. 5

(d) Discuss the role of inverse problem to Magnetic Resonance Imaging (MRI). 5

Unit - III

4. Answer *any one* question :

(a) An X-ray beam $A(x)$ propagates in a medium is defined by $A(x) = n\tanh\alpha x$, $-\infty < x < \infty$, n, $\alpha > 0$. Prove that the intensity of this beam is maximum at the origin and goes to zero as $|x| \to \infty$. 3+2

(b) (i) Explain the parametrization

$$l_{t,\theta} = \{(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta) : -\infty < s < \infty\}.$$

(ii) Find the values of t and θ for which the line $l_{t,\theta}$ is the same as the line with the equation $\sqrt{3}x + y = 4$. 3+2

Unit - IV

5. Answer *any one* question :

(a) If exists, find Radon transform of the function $f(x,y) = \begin{cases} a^2 - \sqrt{x^2+y^2}, & \text{if } x^2+y^2 \le a^2 \\ 0, & \text{if } x^2+y^2 > a^2 \end{cases}$ on a line $\mathcal{L}_{t,\theta}$. 5

(b) (i) Define Radon transform.

(ii) Show that Radon transform R maps a linear combination of functions to the same linear combination of the Radon transforms of the functions separately. 2+3

Unit - V

6. Answer *any one* question :

(a) What is Back Projection? Give a suitable example of Back Projection in medical imaging. 3+2

(b) Given, $f(x,y) = \begin{cases} 1 - \sqrt{x^2+y^2}, & \text{if } x^2+y^2 \le 1 \\ 0, & \text{if } x^2+y^2 > 1 \end{cases}.$

Find the Back Projection of the Radon transform $\mathcal{R}f(t, \theta)$ of f. 5

Unit - VI

7. Answer *any two* questions :

(a) Find the locus of (x, y) such that the back projection of $h(t, \theta) = t^3 \sin\theta$ at the point (x, y) is equal to zero. 5

(b) If $F(\alpha)$ is the Fourier transform of $f(x)$, then prove that the Fourier transform of $\cos(ax)f(x)$ is

$$\frac{1}{2}[F(\alpha - a) + F(\alpha + a)].$$ 5

(c) Describe Kaczmarz's method to find an approximate solution to a linear system $A\tilde{x} = \tilde{b}$. 5

(d) Define Affine spaces and Affine projection. 2½+2½