2024

PHYSICS — HONOURS

Paper : DSE-A-1.1 and DSE-A-1.2

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Paper : DSE-A-1.1

(Advanced Mathematical Methods)

Full Marks : 65

Answer ***question nos.*** **1 and 2**, and ***any four*** questions from the rest.

1. Answer ***any five*** questions : 2×5

(a) Define a binary composition '*' on the real numbers by : $a * b = a^b$. Is this composition associative? Is it commutative?

(b) Show that the projection map 'f' from the vector space R^n to the vector space R given by $f\colon (\xi_1, \xi_2, \ldots, \xi_n) \to \xi_1$ gives a homomorphism from R^n to R.

(c) Show that under orthogonal transformation length of a three vector in Euclidean space remains invariant.

(d) Show that the unit element of a group is unique and the inverse to every element of a group is unique.

(e) $u = (5, 4, 1)$, $v = (3, -4, 1)$ and $w = (1, -2, 3)$. Which pair of vectors if any, are orthogonal?

(f) If R be a relation from A to B, then show that the domain of R is the range of R^{-1} and the range of R is the domain of R^{-1}.

(g) Show that Kronecker delta δ^i_j is a mixed tensor of rank two.

(h) Show that $\sum_{i,j} a_{ij} x_i x_j = 0$ if a_{ij} is anti-symmetric tensor and x_i's are the co-ordinate vector.

2. Answer ***any three*** questions :

(a) (i) Consider the set :

$$S = \{a - b\sqrt{2} \mid a, b \in Z\},$$

where Z is the set of all integers including zero. Show that S forms an abelian group with respect to ordinary addition. Does S forms a field with respect to ordinary addition and ordinary multiplication?

(ii) Show that the set of triplets : $A = \{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in R\}$, where R is the set of reals does not form a vector space if the set of complex numbers is chosen as the set of scalars.

(iii) Show that $f(x) = 2x + 3$ is not a linear transformation from $\mathbb{R}$ to $\mathbb{R}$. (1+2)+1+1

(b) (i) Show that contraction of a 2nd rank tensor yields a scalar.

(ii) Let $a_1, \ldots, a_n$ be a set of linearly dependent vectors. Show that one can find a vector a_k from the given set so that it can be written as linear combination of $a_1, a_2, \ldots, a_{k-1}$. 2+3

(c) (i) Consider Z_4, the group of fourth roots of unity. Construct the group multiplication table with respect to ordinary multiplication. Use the multiplication table to find a subgroup.

(ii) For an Abelian group, show that conjugacy class of any element contains only that element. 4+1

(d) Prove Cauchy-Schwarz inequality for a finite dimensional inner product space V. 5

(e) (i) State Lagrange's theorem in group theory.

(ii) Consider a group whose order is a prime. Prove that the group must be cyclic. 2+3

3. (a) Consider the linear map $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x, y) = (3x + 4y, 2x - 5y)$. Consider basis $E = \{e_1, e_2\} = \{(1, 0), (0, 1)\}$ of $\mathbb{R}^2$. Find the matrix presentation of F in the basis E.

(b) Show that the set of linear operators on a vector space forms a vector space over the same field under point-wise addition and multiplication by scalars.

(c) Let $V = \{a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_i \in R\}$ be the vector space of all real polynomials of degree 3 or less and consider the linear operator D defined by :

$D(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + 2a_2x + 3a_3x^2$.

Find the matrix that represents D with respect to the basis :

$|\alpha_1\rangle = 1,\ |\alpha_2\rangle = x,\ |\alpha_3\rangle = x^2$ and $|\alpha_4\rangle = x^3$ 4+2+4

4. (a) Consider the sub space V of R^4 spanned by the vectors : $v_1 = (1, 1, 1, 1)$, $v_2 = (1, 1, 2, 4)$, $v_3 = (1, 2, -4, -3)$. Apply Gram-Schmidt algorithm to find an orthogonal and orthonormal basis for V.

(b) (i) Find the eigenvalues and corresponding eigenvectors for $A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$.

(ii) Find the diagonal matrix D for the given A. (4+1)+(4+1)

5. (a) An anti-symmetric tensor $F_{\mu\gamma}$ satisfies the equation $\partial_\mu F^{\mu\gamma} = j^\gamma$.

Show that (i) $\partial_\gamma j^\gamma = 0$ (ii) $\partial_\mu A^\mu = \partial^\mu A_\mu$.

(b) (i) Write the Lorentz invariant form of Maxwell's equations in terms of electromagnetic field tensor $F_{\mu\gamma}$.

(ii) From the conservation of four-momentum, show that $E^2 = m_0{}^2c^4 + p^2c^2$, where $\vec{p}$ is the three-momentum. (2+3)+(2½+2½)

6. (a) Define isotropic tensor with example. Using Levi-Civita symbol and summation convention, show that $\vec{A}\times(\vec{B}\times\vec{C})=(\vec{A}\cdot\vec{C})\vec{B}-(\vec{A}\cdot\vec{B})\vec{C}$.

(b) (i) Obtain the expression of different components of inertia tensor for a particle of mass m situated at (a_1, a_2, a_3).

(ii) Show that it is a symmetric tensor. (2+3)+(4+1)

7. (a) Define cyclic group. Consider the group of all integer numbers with '+' as group multiplication rule $(Z, +)$. Identify in detail, the two generators of this group.

(b) Consider a general group which can be represented by $\langle a,b \,|\, a^2=e, b^2=e, ab=ba\rangle$ with a and b being two generators of the group followed by relations among them. Construct a full multiplication table for the group.

(c) (i) For group homomorphism $\phi: G\to G'$, let, e and e' be the respective identity elements. Show that :

$\phi(e)=e'$

$\phi(g^{-1})=(\phi(g))^{-1}$ for any $g\in G$.

(ii) Show that from $\mathbb{Z}_2\to\mathbb{Z}$ there can be only one group homomorphism which is the zero homomorphism. (2+1+1)+3+(2+1)

8. (a) Consider the set of transformations on the points $x\in R$ given by :

$f(\alpha_1, \alpha_2; x) = \alpha_1 x+\alpha_2$, where $\alpha_1, \alpha_2\in R$, $\alpha_1\neq 0$.

Express the transformation in matrix form by using the column matrix $(x, 1)^T$. Obtain the composition matrix for two such transformations $f(\beta_1, \beta_2; x)$ and $f(\alpha_1, \alpha_2; x)$. Hence show that the above set of transformations forms a continuous group.

(b) In the above problem (a), obtain the expressions of generators in matrix form.

(c) Show that the intersection of two groups is another group.

(d) Give geometric interpretation of the members of C_3, the symmetry group of equilateral triangles.

(e) Does set of all 3×3 matrices over real number of non-zero determinant form a group?— Explain. What happens if we include also the matrices of zero determinant? (1+1+1)+2+2+2+1

Paper : DSE-A-1.2

(Laser and Fibre Optics)

Full Marks : 65

Answer *question nos.* 1 and 2, and *any four* questions from the rest.

1. Answer *any five* questions : 2×5

   (a) In a three-level laser system $A_{32} = 10^7 S^{-1}$, $A_{31} = 8 \times 10^7 S^{-1}$ and $A_{21} = 2 \times 10^8 S^{-1}$. Can laser transitions be easily obtained between 2 and 1? What is the spontaneous lifetime in level 3?

   (b) What is Pockel's effect?

   (c) Find the relative population of two states in Ruby laser that produces light beam of wavelength 6943 Å at 300 K.

   (d) Why does a three-level laser normally provide pulsed output?

   (e) The optical power after propagating through a fiber of length 450 m is reduced to 30% of its original value. Calculate the fiber loss in dB/km.

   (f) Calculate the gap in frequency between two longitudinal modes in a laser cavity of length 50 cm and refractive index of medium inside cavity is 1.25.

   (g) Write the differences between linear and non-linear optics.

2. Answer *any three* questions :

   (a) (i) Draw a generic block diagram of a fiber optical communication system. What is the speciality of guided mode?

   (ii) What are the advantages of using optical fiber sensors? (2+1)+2

   (b) An optical resonator with length L has two mirrors of radii of curvature r_1, r_2 respectively. Write down the equation for stability condition of the resonator. Draw the stability diagram and explain. Indicate the points in the stability diagram for the following configurations :

   (i) $r_1 = r_2 = L$ (ii) $r_1 = r_2 = \infty$. 1+2+2

   (c) What is metastable state?

   Show with schematic diagram how laser emission takes place in case of three-level laser system. Write the rate equations for three-level system with suitable explanation. 2+1+2

   (d) Write the differences between step index and graded index optical fiber. A step index optical fiber has a core refractive index $n_1 = 1.48$ and cladding refractive index $n_2 = 1.46$. Determine the maximum acceptance angle $\emptyset_{max}$ of the fiber in air and in water. 2+3

(e) The electric field associated with a mode is given by $E(t) = E_o\ e^{-\omega_0 t/2Q}\ e^{2\pi i \nu_0 t}$.

(i) Find out the frequency spectrum associated with this wave train which extends from $t = 0$ to $t = \infty$.

(ii) Draw frequency dependence curve w.r.t. intensity and indicate FWHM. 2+(2+1)

3. (a) Consider a laser system with Mirror M_1 and M_2 having reflectivity R_1 and R_2 respectively. Mirrors are separated by a distance L.

(i) Write down the beam intensity at M_2 if the intensity at M_1 is I_0 (given γ is the gain coefficient, α is the absorption coefficient).

(ii) Write down the beam intensity after reflection at M_2.

(iii) Write down the final intensity after completing one round trip.

(iv) Find out the amplification factor.

(v) Find out the condition for lasing action and write down the threshold gain.

(b) An injection laser has active cavity with losses of 30/cm and reflectivity of each mirror is 30%. Determine the laser gain coefficient (per cm unit) for the cavity with length 600 cm.

(1+1+2+1+3)+2

4. (a) What is holography?

(b) What role does laser play in holography?

(c) Explain the basic principle of recording a hologram.

(d) What is Optical Parametric Amplification (OPA)? 2+2+4+2

5. (a) Explain the concept of coherence. Discuss temporal and spatial coherence in case of laser light.

(b) What is monochromaticity? Discuss monochromaticity for laser light. The coherence time for the red cadmium line (6438 Å) is about 10^{-9} sec. Estimate the monochromaticity of the line.

(1+2+2)+(1+2+2)

6. (a) What is optical waveguide? Explain with schematic diagram, step index waveguide and parabolic waveguide.

(b) Derive the one-dimensional ray equation in x-z plane in an optical medium for which $n = n(x)$. What will be the equation for homogeneous medium? (1+2+2)+(4+1)

7. (a) Find an expression for the intensity of Second Harmonic Generation at the exit surface of a material.

(b) From the expression for the intensity, obtain the criterion for pulse matching.

(c) Why is it called refractive index criteria? 5+3+2

8. (a) With proper schematic diagram, deduce the relation between Einstein's A, B coefficient. Hence at thermal equilibrium, obtain the ratio of spontaneous emission and stimulated emission.

(b) Show that the spontaneous emission is more predominant than stimulated emission in optical region. (consider $T = 1000$ K) 5+2+3

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